CHAPTER 2

COMPLEX NUMBERS

Imaginary Numbers: the Symbol i

The square root of a negative number is called an **imaginary** number, e.g. $\sqrt{-4}, \sqrt{-9}, \sqrt{-64}, \sqrt{-100}$ are imaginary numbers.

Imaginary numbers cannot be represented by a real number, as there is no real number whose square is a negative number.

To overcome this problem, the letter i is introduced to represent $\sqrt{-1}$.

$$i = \sqrt{-1}$$

All imaginary numbers can now be expressed in terms of i, for example:

$$\sqrt{-36} = \sqrt{36 \times -1} = \sqrt{36}\sqrt{-1} = 6i$$

$$\sqrt{-50} = \sqrt{50 \times -1} = \sqrt{25 \times 2 \times -1} = \sqrt{25}\,\sqrt{2}\,\sqrt{-1} = 5\sqrt{2}i$$

Integer Powers of i

Every integer power of i is a member of the set $\{1, -1, i, -i\}$.

$$i = \sqrt{-1}$$
$$i^2 = -1$$
$$i^3 = i^2 \,.\, i = (-1)i = -i$$
$$i^4 = i^2 \,.\, i^2 = (-1)(-1) = 1$$

$$i = \sqrt{-1}$$
$$i^2 = -1$$
$$i^3 = -i$$
$$i^4 = 1$$

Example

Simplify: **(i)** i^{21} **(ii)** i^{10} **(iii)** i^{-13}.

Solution:

(i) i^{21}

$= i^4 . i^4 . i^4 . i^4 . i^4 . i$

$= (1)(1)(1)(1)(1)i$

$= i$

Alternatively,

$i^{21} = i^{20} . i$

$= (i^4)^5 . i$

$= (1)^5 . i = i$

(ii) i^{10}

$= i^{10}$

$= i^4 . i^4 . i^2$

$= (1)(1)(-1)$

$= -1$

Alternatively,

$i^{10} = i^8 . i^2$

$= (i^4)^2 . i^2$

$= (1)^2 . (-1)$

$= -1$

(iii) i^{-13}

$= \frac{1}{i^{13}}$

$= \frac{1}{i^{13}} \times \frac{i^3}{i^3}$

$= \frac{i^3}{i^{16}}$

$= \frac{i^3}{1}$

$= i^3 = -i$

Complex Numbers

A complex number has two parts, a **real** part and an **imaginary** part.
Some examples are $3+4i$, $2-5i$, $-6+0i$, $0-i$.
Consider the complex number $4+3i$:

4 is called the **real** part,
3 is called the **imaginary** part.

Note: $3i$ is **not** the imaginary part.

> Complex number = (Real Part) + (Imaginary Part) i

The set of complex numbers is denoted by ***C***.
The letter z is usually used to represent a complex number, e.g.

$z_1 = 2+3i$, $z_2 = -2-i$, $z_3 = -5i$

If $z = a + bi$, then:

(i) a is called the real part of z and is written $\mathbf{Re}(z) = a$

(ii) b is called the imaginary part of z and is written $\mathbf{Im}(z) = b$.

Example

Write down the real and imaginary parts of each of the following complex numbers:

(i) $5-4i$ **(ii)** $-3+2i$ **(iii)** 6 **(iv)** $-5i$.

Solution:

	Real Part	**Imaginary Part**
(i) $5-4i$	5	−4
(ii) $-3+2i$	−3	2
(iii) $6=6+0i$	6	0
(iv) $-5i=0-5i$	0	−5

Notes: i **never** appears in the imaginary part.
If $Im(z)=0$, then z is a **real** number.
If $Re(z)=0$, then z is a **purely imaginary** number.

Addition and Subtraction of Complex Numbers

To add or subtract complex numbers do the following:

Add or subtract the real and the imaginary parts separately.

For example:

$$(3+5i)-(2-3i)=3+5i-2+3i=1+8i$$

$$2(-1+3i)-3(1+4i)=-2+6i-3-12i=-5-6i$$

Multiplication of Complex Numbers

Multiplication of complex numbers is performed using the usual algebraic method, except that:

i^2 is replaced with −1.

For example:

$$\begin{aligned}(3-2i)(-4+5i) &= 3(-4+5i)-2i(-4+5i)\\ &=-12+15i+8i-10i^2\\ &=-12+15i+8i-10(-1) \qquad (i^2=-1)\\ &=-12+15i+8i+10\\ &=-2+23i\end{aligned}$$

Example

If $z_1 = 3 + 2i$ and $z_2 = -1 + 5i$, express in the form $a + bi$, $a, b \in \mathbf{R}$:

(i) $2z_1 - iz_2$ **(ii)** $z_1 z_2$ **(iii)** z_1^2.

Solution:

(i) $2z_1 - iz_2$

$= 2(3 + 2i) - i(-1 + 5i)$

$= 6 + 4i + i - 5i^2$

$= 6 + 4i + i + 5$

$(i^2 = -1)$

$= 11 + 5i$

(ii) $z_1 z_2$

$= (3 + 2i)(-1 + 5i)$

$= 3(-1 + 5i) + 2i(-1 + 5i)$

$= -3 + 15i - 2i + 10i^2$

$= -3 + 15i - 2i - 10$ $(i^2 = -1)$

$= -13 + 13i$

(iii) z_1^2

$= (3 + 2i)^2$

$= (3 + 2i)(3 + 2i)$

$= 3(3 + 2i) + 2i(3 + 2i)$

$= 9 + 6i + 6i + 4i^2$

$= 9 + 6i + 6i - 4$ $(i^2 = -1)$

$= 5 + 12i$

Example

Given that $z = 1 - \sqrt{3}i$, find the real number k such that $z^2 + kz$ is:

(i) real (ii) purely imaginary.

Solution:

$z = 1 - \sqrt{3}i$

$z^2 = (1 - \sqrt{3}i)^2$

$= (1 - \sqrt{3}i)(1 - \sqrt{3}i)$

$= 1(1 - \sqrt{3}i) - \sqrt{3}i(1 - \sqrt{3}i)$

$= 1 - \sqrt{3}i - \sqrt{3}i + 3i^2$

$= 1 - \sqrt{3}i - \sqrt{3}i - 3$

$= -2 - 2\sqrt{3}i$

$z^2 + kz$

$= (-2 - 2\sqrt{3}i) + k(1 - \sqrt{3}i)$

$= -2 - 2\sqrt{3}i + k - k\sqrt{3}i$

$= (-2 + k) + (-2\sqrt{3} - k\sqrt{3})i$

(Group real and imaginary parts together)

(i) If $z^2 + kz$ is real, then the imaginary part is zero,

$\therefore \quad -2\sqrt{3} - k\sqrt{3} = 0$

$-2 - k = 0$

$k = -2$

(ii) If $z^2 + kz$ is imaginary, then the real part is zero,

$\therefore \quad -2 + k = 0$

$k = 2$

Note: $\sqrt{3}i$ is often written as $i\sqrt{3}$ to avoid the error $\sqrt{3i}$.

Exercise 2.1

Express each of the following in the form ai, where $a \in \mathbf{N}$:

1. $\sqrt{-4}$ **2.** $\sqrt{-25}$ **3.** $\sqrt{-49}$ **4.** $\sqrt{-100}$ **5.** $\sqrt{-16}$ **6.** $\sqrt{-144}$

7. $\sqrt{-9}$ **8.** $\sqrt{-121}$ **9.** $\sqrt{-400}$ **10.** $\sqrt{-196}$ **11.** $\sqrt{-169}$ **12.** $\sqrt{-289}$

Express each of the following in the form $a\sqrt{b}i$, $a, b \in \mathbf{N}$ and b is prime:

13. $\sqrt{-8}$ **14.** $\sqrt{-12}$ **15.** $\sqrt{-18}$ **16.** $\sqrt{-50}$ **17.** $\sqrt{-80}$ **18.** $\sqrt{-63}$

Express each of the following as an element of the set $\{1, -1, i, -i\}$:

19. i^6 **20.** i^7 **21.** i^8 **22.** i^{13} **23.** i^{20} **24.** i^{22}

25. i^{27} **26.** i^{102} **27.** i^{-3} **28.** i^{-2} **29.** i^{-1} **30.** i^{-20}

31. Write down the real part and imaginary parts of z if:

(i) $z = 2 + 5i$ **(ii)** $z = -3 - 4i$ **(iii)** $z = -7 - \sqrt{3}i$ **(iv)** $z = -7i$

Express each of the following in the form $a + bi$, $a, b \in \mathbf{R}$ and $i^2 = -1$:

32. $(3 + 2i) - 2(5 - 4i)$

33. $2(2 - 3i) - 3(2 - 7i)$

34. $(2 + 3i)(-3 + 4i)$

35. $(3 + i)(-2 - 5i)$

36. $3 + 2i(3 + 4i) - i$

37. $i(-2 + 5i) - 5(-1 + 2i)$

38. $i(3 - 5i)(4 + i)$

39. $4i(2 - 3i)(-2 - 4i)$

40. $(1 + 2i)^2 - 2(5 - 3i)$

41. $(1 + \sqrt{3}i)^2 - \sqrt{3}(-\sqrt{3} + 2i)$

42. If $z = 1 - 3i$, where $i^2 = -1$, evaluate $z^2 - 3z$.

43. If $z_1 = 5 - 2i$, $z_2 = -2 - 3i$, express in the form $a + bi$, $a, b \in \mathbf{R}$ and $i^2 = -1$:
(i) $z_1 z_2$ **(ii)** z_2^2 **(iii)** iz_1 **(iv)** $iz_1 z_2$.

44. If $w = 2 + 3i$, show that $w^2 - 4w + 13 = 0$.

45. Find the value of $k \in \mathbf{R}$ if $(1 - 3i)(k + 2i)$ is real, where $i^2 = -1$.

46. Find the value of $k \in \mathbf{R}$ if $(2 + i)(k - 5i)$ is purely imaginary, where $i^2 = -1$.

47. Given that $z = 1 + 3i$, find the real number k such that $z^2 + kz$ is real.

48. Given that $z = 1 + \sqrt{2}i$, find the real number t such that $z^2 + tz$ is:
(i) real **(ii)** purely imaginary.

49. Given that $z = (-1 + \sqrt{5}i)$, find the real number k such that $z^2 - kz + \sqrt{5}i$ is:
(i) real **(ii)** purely imaginary.

Conjugate and Division

Conjugate of a Complex Number

Two complex numbers which differ only in the sign of their imaginary parts are called **conjugate complex numbers**, each being the conjugate of the other.

Thus $3+4i$ and $3-4i$ are conjugates, and $-2-3i$ is the conjugate of $-2+3i$ and vice versa.

In general, $a+bi$ and $a-bi$ are conjugates.

If $z=a+bi$, then its conjugate, $a-bi$, is denoted by $\bar{z}$.

$$z=a+bi \quad \Rightarrow \quad \bar{z}=a-bi$$

To find the conjugate, simply **change the sign** of the imaginary part only.

For example, if $z=-6-5i$ then $\bar{z}=-6+5i$.

Example

If $z=-3+5i$, where $i^2=-1$, simplify: **(i)** $z+\bar{z}$ **(ii)** $z-\bar{z}$ **(iii)** $z\bar{z}$.

Solution:

If $z=-3+5i$, then $\bar{z}=-3-5i$ (change sign of the imaginary part only).

(i) $z+\bar{z}$
$=(-3+5i)+(-3-5i)$
$=-3+5i-3-5i$
$=-6$ (a real number)

(ii) $z-\bar{z}$
$=(-3+5i)-(-3-5i)$
$=-3+5i+3+5i$
$=10i$ (a purely imaginary number)

(iii) $z\bar{z}$
$=(-3+5i)(-3-5i)$
$=-3(-3-5i)+5i(-3-5i)$
$=9+\cancel{15i}-\cancel{15i}-25i^2$
$=9+25 \quad (i^2=-1)$
$=34$ (a real number)

Note: If a complex number is added to, or multiplied by, its conjugate the imaginary parts cancel and the result will **always** be a real number.

If $z=a+bi$, then:

1. $z+\bar{z}=(a+bi)+(a-bi)=a+bi+a-bi=2a$ (a real number)
2. $z\bar{z}=(a+bi)(a-bi)=a^2-abi+abi-b^2i^2=a^2+b^2$ (a real number)

Division by a Complex Number

Multiply the top and bottom by the conjugate of the bottom.

This will convert the complex number on the bottom into a real number. The division is then performed by dividing the real number on the bottom into each part on the top.

Example

If $z = \dfrac{2+i}{1-i}$, find the real part of z.

Solution:

First write z in the form $a+bi$:

$$z = \frac{2+i}{1-i}$$

$$= \frac{2+i}{1-i} \cdot \frac{1+i}{1-i} \qquad \left(\text{multiply the top and bottom by } 1+i, \text{ the conjugate of } 1-i\right)$$

$$= \frac{2+2i+i+i^2}{1+i-i-i^2}$$

$$= \frac{2+2i+i-1}{1+i-i+1} \qquad (i^2=-1)$$

$$= \frac{1+3i}{2}$$

$$= \tfrac{1}{2} + \tfrac{3}{2}i \qquad \text{(divide the bottom into } \textbf{each} \text{ part on top)}$$

Thus, the real part of z is $\frac{1}{2}$.

Example

Simplify $\dfrac{4+3i}{3-4i}$ and, hence, evaluate $\left(\dfrac{4+3i}{3-4i}\right)^{10}$.

Solution:

$$\frac{4+3i}{3-4i}$$

$$= \frac{4+3i}{3-4i} \cdot \frac{3+4i}{3+4i}$$

$$= \frac{12+16i+9i+12i^2}{9+12i-12i-16i^2}$$

$$= \frac{25i}{25} \qquad (i^2=-1)$$

$$= i$$

$$\frac{4+3i}{3-4i} = i$$

$$\therefore \quad \left(\frac{4+3i}{3-4i}\right)^{10} = i^{10}$$

$$= i^4 \cdot i^4 \cdot i^2$$

$$= (1)(1)(-1)$$

$$= -1$$

Exercise 2.2

Express each of the following in the form $a+bi$, where $a, b \in \mathbf{R}$ and $i^2=-1$:

1. $\dfrac{3+4i}{2+i}$
2. $\dfrac{7+4i}{2-i}$
3. $\dfrac{1+5i}{3+2i}$
4. $\dfrac{7-17i}{5-i}$
5. $\dfrac{1}{1-i}$
6. $\dfrac{2+i}{1+2i}$
7. $\dfrac{2-i}{3+2i}$
8. $\dfrac{3+4i}{1-i}$

9. If $a + bi = \frac{9 - 7i}{2 - 3i}$, find the value of a and the value of b, $a, b \in \mathbf{R}$.

10. If $p + qi = \frac{2 - i}{1 - 2i}$, $p, q \in \mathbf{R}$, evaluate $p^2 + q^2$.

11. Given that $(4 + 3i)z = 1 + 7i$, express the complex number z in the form $a + bi$.

12. (i) Express $(1 - 2i)^2$ in the form $a + bi$.
(ii) Hence, find the real part of $\frac{1}{(1 - 2i)^2}$.

13. Show that of $\frac{1}{(1 + i)^2}$ is a purely imaginary number and write down the imaginary part.

14. Evaluate: $\left(\frac{1}{2 + i} + \frac{2}{3 - i}\right)^{100}$.

15. (i) Evaluate $\frac{a + bi}{b - ai}$, where $a, b \in \mathbf{R}$ and $i^2 = -1$.
Hence, or otherwise, evaluate:
(ii) $\left(\frac{5 + 2i}{2 - 5i}\right)^4$ (iii) $\left(\frac{3 + 4i}{4 - 3i}\right)^4$ (iv) $\left(\frac{-3 + 2i}{2 + 3i}\right)^{21}$ (v) $\left(\frac{2 + i}{1 - 2i}\right)^{31}$

Evaluate each of the following:

16. $\left(\frac{8 - 4i}{2 - i}\right)^3$ 17. $\left(\frac{-6 + 8i}{4 + 3i}\right)^8$ 18. $\left(\frac{2 - 3i}{9 + 6i}\right)^4$ 19. $\left(\frac{-1 + \sqrt{2}i}{\sqrt{2} + i}\right)^{34}$

20. k is a real number such that $\frac{-1 + \sqrt{3}i}{-5\sqrt{3} - 5i} = ki$. Find k.

Equality of Complex Numbers

If two complex numbers are equal then:
their real parts are equal and their imaginary parts are also equal.

For example, if $a + bi = c + di$,

then $a = c$ and $b = d$.

This definition is very useful when dealing with equations involving complex numbers.

Equations involving complex numbers are usually solved with the following steps:

1. Remove the brackets.
2. Put an R under the real parts and an I under the imaginary parts to identify them.
3. Let the real parts equal the real parts and the imaginary parts equal the imaginary parts.
4. Solve these resultant equations (usually simultaneous equations).

Note: If one side of the equation does not contain a real part or an imaginary part, it should be replaced with 0 or $0i$, respectively.

Example

$z_1 = 4 - 2i$, $z_2 = -2 - 6i$. If $z_2 - pz_1 = qi$, $p, q \in \mathbf{R}$, find p and q.

Solution:

$z_2 - pz_1 = qi$

The right-hand side has no real part, hence a 0, representing the real part, should be placed on the right-hand side.

Now the equation is:

$z_2 - pz_1 = 0 + qi$ (put 0 in for real part)

$(-2-6i) - p(4-2i) = 0 + qi$ (substitute for z_1 and z_2)

$-2 - 6i - 4p + 2pi = 0 + qi$ (remove the brackets)

$R \quad I \quad R \quad I \quad R \quad I$ (identify real and imaginary parts)

Real parts = Real parts

$-2 - 4p = 0$ .... ①

Imaginary parts = Imaginary parts

$-6 + 2p = q$ .... ②

Solve between the equations ① and ②:

$-2 - 4p = 0$ ①

$-4p = 2$

$4p = -2$

$p = -\frac{2}{4} = -\frac{1}{2}$

Substitute $p = -\frac{1}{2}$ into equation ②:

$-6 + 2p = q$ ②

$-6 + 2(-\frac{1}{2}) = q$

$-6 - 1 = q$

$-7 = q$

Solution: $p = -\frac{1}{2}$, $q = -7$

Example

$w = a + bi$ is a complex number such that: $w\overline{w} - 2iw = 17 - 6i$.

Find the two possible values of w.

Solution:

$w = a + bi \quad \therefore \quad \overline{w} = a - bi$

Given: $w\overline{w} - 2iw = 17 - 6i$

$\therefore \quad (a+bi)(a-bi) - 2i(a+bi) = 17 - 6i$

$a^2 - \cancel{abi} + \cancel{abi} - b^2i^2 - 2ai - 2bi^2 = 17 - 6i$ (remove brackets)

$a^2 + b^2 - 2ai + 2b = 17 - 6i$ $(i^2 = -1)$

$R \quad R \quad I \quad R \quad R \quad I$ (identify real and imaginary parts)

Real parts = Real parts

$a^2 + b^2 + 2b = 17$ ①

Imaginary parts = Imaginary parts

$-2a = -6$ ②

Solve between equations ① and ②:

$$-2a = -6 \quad ②$$
$$2a = 6$$
$$a = 3$$

$$a^2 + b^2 + 2b = 17$$
$$9 + b^2 + 2b = 17 \quad (a = 3)$$
$$b^2 + 2b - 8 = 0$$
$$(b+4)(b-2) = 0$$
$$b = -4 \quad \text{or} \quad b = 2$$

$w = a + bi$

Thus, $w = 3 - 4i$ or $w = 3 + 2i$.

Exercise 2.3

Solve for $x, y \in \mathbf{R}$:

1. $x(3+4i) + y(2-3i) = 8 + 5i$
2. $x(3-2i) + y(i-2) = 5 - 4i$
3. $3x - i(x+y+5) = (1+3i)i + 2(3-y)$
4. $(x+y) - (xi - y) = (3+2i)^2 - 7i$
5. If $2(h-2) - k + i = i(2k - h)$, find h and k, $h, k \in \mathbf{R}$.
6. If $2p - q + i(7i+3) = 2(2i - q) - i(p + 3q)$, find p and q, $p, q \in \mathbf{R}$.
7. $z_1 = 4 - 3i$, $z_2 = 5(1+i)$. If $z_1 + tz_2 = k$, find t and k, $t, k \in \mathbf{R}$.
8. $z_1 = 5 + 7i$, $z_2 = 3 - i$. If $k(z_1 + z_2) = 16 + (t+2)i$, find t and k, $t, k \in \mathbf{R}$.
9. $z = 2 - 3i$. If $z + i + 3(a + bi) = iz - 5$, find a and b, $a, b \in \mathbf{R}$.
10. $z_1 = 2 + 3i$, $z_2 = -4 - 3i$. If $lz_1 - z_2 = ki$, find l and k, $l, k \in \mathbf{R}$.
11. $z_1 = 6 - 8i$, $z_2 = 4 - 3i$. If $pi = z_2 + lz_3$, find p and l, where $z_1 - z_3 = z_2$ and $p, l \in \mathbf{R}$.

Express z in the form $a + bi$ if:

12. $iz = 2(3 - \bar{z})$
13. $z(1+3i) - 5(1+3i) = 2z$
14. $z = a + bi$ is a complex number such that $z + \bar{z} = 8$ and $z\bar{z} = 25$.
    Find two values of z.
15. $w = a + bi$ is a complex number such that $w\bar{w} - 3iw = 5(7 - 3i)$.
    Find two values of w.
16. $z = p + qi$ is a complex number such that $z\bar{z} - i\bar{z} = 11 - 3i$.
    Find two values of z.
17. If $a(a+i) - bi(3 + bi) = 10(1+i)$, find a, b, $a, b \in \mathbf{R}$.
18. If $a^2 + 2abi - b^2 = -15 + 8i$, find the values of a, b, $a, b \in \mathbf{R}$.

Argand Diagram and Modulus

Argand Diagram

An Argand diagram is used to plot complex numbers. It is very similar to the x- and y-axes used in coordinate geometry, except that the **horizontal** axis is called the **real axis (Re)** and the **vertical** axis is called the imaginary axis (**Im**). It is also called the **complex plane**.

To represent a complex number on an Argand diagram, it must be written in the form $a+bi$.

The complex number $a+bi$ is represented by the point with coordinates (a, b).

Example

If $z_1 = 2-3i$ and $z_2 = 6-5i$, represent on an Argand diagram:

(i) $\bar{z}_1$ **(ii)** $2z_1 - z_2$.

Solution:

(i) $z_1 = 2-3i$

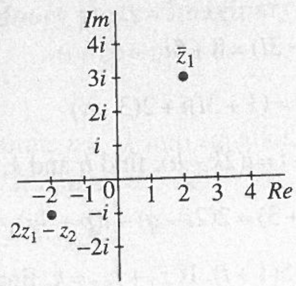
$\bar{z}_1 = 2+3i$

(ii) $2z_1 - z_2$

$= 2(2-3i)-(6-5i)$

$= 4-6i-6+5i$

$= -2-i$



Modulus of a Complex Number

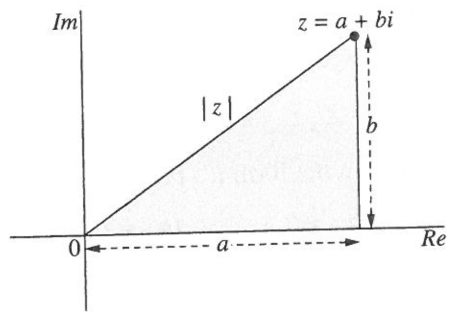
The **modulus** of a complex number is the distance from the origin to the point representing the complex number on the Argand diagram.

If $z = a+bi$, then the modulus of z is written $|z|$ or $|a+bi|$.

The point z represents the complex number $a+bi$.

The modulus of z is the distance from the origin, o, to the complex number $a+bi$.

Using the theorem of Pythagoras, $|z| = \sqrt{a^2+b^2}$.



If $z = a+bi$, then

$|z| = |a+bi| = \sqrt{a^2+b^2}$.

Notes:

1. i **never** appears when the modulus formula is used.
2. The modulus of a complex number is **always positive**.
3. Before using the formula a complex number must be in the form $a+bi$.

For example, if $z = -2+5i$, then:

$|z| = |-2+5i| = \sqrt{(-2)^2+(5)^2} = \sqrt{4+25} = \sqrt{29}$

Example

Let $z = (k-1) + 7i$ and $w = 8 - i$.

If $|z| = |w|$, find two values of k, $k \in \mathbf{R}$.

Given: $|z| = |w|$

$\therefore$

$$|(k-1)+7i| = |8-i|$$

$$\sqrt{(k-1)^2+(7)^2} = \sqrt{8^2+(-1)^2} \qquad \left(|a+bi| = \sqrt{a^2+b^2}\right)$$

$$(k-1)^2+(7)^2 = 8^2+(-1)^2 \qquad \text{(square both sides)}$$

$$k^2-2k+1+49 = 64+1$$

$$k^2-2k-15=0$$

$$(k-5)(k+3)=0$$

$$k=5 \quad \text{or} \quad k=-3$$

Exercise 2.4

For questions 1 – 11, construct an Argand diagram from –6 to 6 on the real axis and $-5i$ to $5i$ on the imaginary axis.

If $z = 1 + i$ and $w = -6 + 4i$, represent each of the following on an Argand diagram:

1. z **2.** $\bar{z}$ **3.** w **4.** $\bar{w}$ **5.** $2\bar{z} + w$

6. $\frac{1}{2}z\bar{w}$ **7.** $\dfrac{w}{z}$ **8.** $\dfrac{\bar{z}-w+3}{\bar{z}+1}$ **9.** $\dfrac{i\bar{w}}{z}$ **10.** $\dfrac{w}{z^2}$

11. Let $w = 9 + 7i$ and $u = \dfrac{5+i}{1-i}$. Represent on an Argand diagram:

(i) u **(ii)** $\dfrac{w}{u}$ **(iii)** $\dfrac{w-4(u-1)}{3-u}$.

Hence, evaluate $\left|\dfrac{w-4(u-1)}{3-u}\right|$.

12. Evaluate each of the following:

(i) $|12-5i|$ **(ii)** $|-3-5i|$ **(iii)** $|\sqrt{2}+i|$ **(iv)** $|1-\sqrt{3}i|$

(v) $|2-2\sqrt{3}i|$ **(vi)** $|-2-\frac{3}{2}i|$ **(vii)** $|-\frac{1}{2}+\frac{\sqrt{3}}{2}i|$ **(viii)** $|\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i|$

13. Let $w = \dfrac{4-2i}{2+i}$.

(i) Express w in the form $a + bi$, where $a, b \in \mathbf{R}$. **(ii)** Evaluate $|w|$.

14. Let $z = 4 - 3i$ and $w = \dfrac{1-\sqrt{3}i}{1+\sqrt{3}i}$. Evaluate: **(i)** $\left|\dfrac{1}{z}\right|$ **(ii)** $|w|$.

15. Let $z = 1 + 7i$ and $w = -1 + i$. Express $\dfrac{z}{w}$ in the form $a + bi$, $a, b \in \mathbf{R}$ and $i^2 = -1$.

Verify that: **(i)** $|z| \cdot |w| = |zw|$ **(ii)** $\dfrac{|z|}{|w|} = \left|\dfrac{z}{w}\right|$.

Solve, for real h and k: $hz = \left|\dfrac{z}{w}\right|kw + 16i$.

16. If $|8+ki|=10$, $k\in\mathbf{R}$, find two possible values of k.

17. If $|a+ai|=|1-7i|$, $a\in\mathbf{R}$, find two possible values of a.

18. Let $z=(k-1)-5i$ and $w=-2+11i$.
If $|z|=|w|$, find two possible values of $k\in\mathbf{R}$.

19. Let $z=a+bi$, where a, $b\in\mathbf{R}$.
If $z-\bar{z}+|z|=17+16i$, find two values of z.

20. Let $w=a+bi$. Find the complex number w such that $\sqrt{3}|w|+iw=2+\sqrt{2}i$.

Quadratic Equations with Complex Roots

The equation $az^2+bz+c=0$ has roots given by:

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

If $b^2-4ac<0$, then the number under the square root sign will be negative, and so the solutions will be complex numbers.

Example

Solve the equations: **(i)** $z^2-4z+13=0$ **(ii)** $2z^2+2z+1=0$.

Solution:

(i) $z^2-4z+13=0$

$az^2+bz+c=0$

$a=1,\quad b=-4,\quad c=13$

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$z=\frac{4\pm\sqrt{(-4)^2-4(1)(13)}}{2(1)}$$

$$z=\frac{4\pm\sqrt{16-52}}{2}$$

$$z=\frac{4\pm\sqrt{-36}}{2}$$

$$z=\frac{4\pm 6i}{2}$$

$$z=2\pm 3i$$

$\therefore$ the roots are $2+3i$ and $2-3i$.

(ii) $2z^2+2z+1=0$

$az^2+bz+c=0$

$a=2,\quad b=2,\quad c=1$

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$z=\frac{-2\pm\sqrt{(2)^2-4(2)(1)}}{2(2)}$$

$$z=\frac{-2\pm\sqrt{4-8}}{4}$$

$$z=\frac{-2\pm\sqrt{-4}}{4}$$

$$z=\frac{-2\pm 2i}{4}=\frac{-1\pm i}{2}$$

$$z=-\tfrac{1}{2}\pm\tfrac{1}{2}i$$

$\therefore$ the roots are $-\frac{1}{2}+\frac{1}{2}i$ and $-\frac{1}{2}-\frac{1}{2}i$.

Note: Notice that in both solutions the roots occur in conjugate pairs. If one root of a quadratic equation, with real coefficients, is a complex number, then the other root must also be complex and the conjugate of the first.

i.e. if $3-4i$ is a root, then $3+4i$ is also a root;
if $-2-5i$ is a root, then $-2+5i$ is also a root;
if $a+bi$ is a root, then $a-bi$ is also a root.

Conjugate Roots Theorem

> If all the coefficients of a polynomial equation are **real**, then all complex roots occur as conjugate pairs.

In other words, if one root is a complex number, then its conjugate is also a root. The Conjugate Roots Theorem can be used only if **all** the coefficients in the equation are **real**. If even one coefficient is non-real (contains an i), then the Conjugate Roots Theorem cannot be used.

The roots of the equation $z^2-2z+10=0$ are $1+3i$ and $1-3i$.
The complex roots occur as conjugate pairs, since all the coefficients, 1, −2 and 10, are real.
The roots of the equation $z^2+(i-2)z+(3-i)=0$ are $1+i$ and $1-2i$.
The complex roots do **not** occur as conjugate pairs, because the coefficients, 1, $i-2$ and $3-i$ are **not** all real numbers.

Example

Solve $z^2-(2-i)z+7-i=0$. Explain why the roots do not occur in conjugate pairs.

Solution:

$$z^2-(2-i)z+7-i=0$$

$$z^2+(-2+i)z+(7-i)=0 \qquad \text{(write in the form } az^2+bz+c=0\text{)}$$

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-2+i)\pm\sqrt{(-2+i)^2-4(1)(7-i)}}{2(1)}$$

$$=\frac{2-i\pm\sqrt{4-4i-1-28+4i}}{2}$$

$$=\frac{2-i\pm\sqrt{-25}}{2}=\frac{2-i\pm 5i}{2}$$

$$z_1=\frac{2-i+5i}{2}=\frac{2+4i}{2}=1+2i$$

$$z_2=\frac{2-i-5i}{2}=\frac{2-6i}{2}=1-3i$$

Thus, the roots are $1+2i$ and $1-3i$.
They are not complex conjugates because **not** all the coefficients are real.

The results concerning the roots of a quadratic equation also hold for quadratic equations that contain complex roots.

If α and β are the roots of the equation $az^2 + bz + c = 0$, then:

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}.$$

The quadratic equation can be written:

$$z^2 - (\alpha + \beta)z + \alpha\beta = 0$$
or
$$z^2 - (\text{sum of the roots})z + (\text{product of the roots}) = 0.$$

Example

If $-2 + 5i$ is a root of the equation $z^2 + pz + q = 0$, $\quad p, q \in \mathbf{R}$, write down the other root and, hence, find the value of p and the value of q.

Solution:

All the coefficients are real (given). Thus we can use the Conjugate Root Theorem.

Therefore, as $-2 + 5i$ is a root, then $-2 - 5i$ is also a root.

$$z^2 - (\text{sum of the roots})z + (\text{product of the roots}) = 0$$

$$z^2 - (-4)z + 29 = 0$$

$$z^2 + 4z + 29 = 0$$

Compare to: $z^2 + pz + q = 0$

$\therefore \quad p = 4 \quad$ and $\quad q = 29$

sum of the roots $= -2 + 5i - 2 - 5i = -4$
product of the roots $= (-2 + 5i)(-2 - 5i) = 29$

Example

Find the value of p and the value of q, $\quad p, q \in \mathbf{R}$, if $(-1 + i)$ is a root of the equation $z^2 + (-1 + pi)z + (q - i) = 0$, and find the other root.

Solution:

$$z^2 + (-1 + pi)z + (q - i) = 0$$

$$(-1 + i)^2 + (-1 + pi)(-1 + i) + (q - i) = 0 \qquad \text{(put in } (-1 + i) \text{ for } z\text{)}$$

$$1 - 2i - 1 + 1 - i - pi - p + q - i = 0$$

$$(-p + q + 1) + (-p - 4)i = 0 \qquad \text{(group real and imaginary parts together)}$$

$\therefore \quad -p + q + 1 = 0 \quad$ ① $\quad$ and $\quad -p - 4 = 0 \quad$ ②

Solving the simultaneous equations ① and ② gives $p=-4$ and $q=-5$.

Thus, $\quad z^2+(-1+pi)z+(q-i)=0$

becomes $\quad z^2+(-1-4i)z+(-5-i)=0 \quad$ (put in $p=-4$ and $q=-5$)

Let $a+bi$ be the other root.

Thus, the roots are $a+bi$ and $-1+i$.

The sum of the roots $=-(-1-4i)=1+4i$

$$\therefore \quad a+bi-1+i=1+4i$$
$$a+bi=1+4i+1-i$$
$$a+bi=2+3i$$

Thus, the other root is $2+3i$.

Exercise 2.5

Solve each of the following equations for $z\in\mathbf{C}$:

1. $z^2-2z+2=0$
2. $z^2-2z+5=0$
3. $z^2+10z+34=0$
4. $2z^2-2z+1=0$
5. $2z^2+6z+5=0$
6. $5z^2-2z+10=0$
7. $z^2-(1+4i)z-2(3-i)=0$
8. $z^2-(3-2i)z-(1+3i)=0$
9. $z^2-(2+5i)z+5(i-1)=0$
10. $z^2-2(1+i)z+(4+2i)=0$

Find a quadratic equation whose roots are:

11. $-2\pm i$
12. $3\pm 2i$
13. $-1\pm 5i$
14. $\pm 3i$
15. $-1\pm\sqrt{2}i$
16. $2\pm\sqrt{5}i$
17. $\frac{1}{2}\pm\frac{1}{2}i$
18. $-\frac{1}{3}\pm\frac{2}{3}i$
19. If $3+5i$ is a root of $z^2+pz+q=0$, $\quad p, q\in\mathbf{R}$, find p and q.
20. Express $\dfrac{1+7i}{1-3i}$ in the form $p+qi$.

    Hence, show that $\dfrac{1+7i}{1-3i}$ is a root of the equation $z^2+4z+5=0$, and write down the other root in the form $a+bi$, $\quad a, b\in\mathbf{R}$.
21. Show that $\dfrac{11-7i}{2+i}$ is a root of the equation $z^2-6z+34=0$ and write down the other root in the form $p+qi$, $\quad p, q\in\mathbf{R}$.
22. If $\dfrac{7-17i}{5-i}$ is a root of $z^2+az+b=0$, $\quad a, b\in\mathbf{R}$, find the values of a and b.
23. If $z=\dfrac{19-4i}{3-2i}=a+bi$, find the value of a and the value of b, $\quad a, b\in\mathbf{R}$.

    Verify that $z^2-10z+29=0$.

    Hence, find two complex numbers, u, such that $(u+3i)^2-10(u+3i)+29=0$.

24. Show that $2+i$ is a root of the equation $z^2-3(1+i)z+5i=0$, and find the other root.

25. Find the value of $k \in \mathbf{R}$ if $1+2i$ is a root of the equation $z^2+kz+7+4i=0$.

26. One root of the equation $z^2+(-1+pi)z+q(2-i)=0$ is $-7i$.
Find: **(i)** the value of p and the value of q **(ii)** the other root.

27. One root of the equation $z^2-(p+i)z+qi=0$ is $2+3i$.
Find the value of p, the value of q and the other root.

28. One root of the equation $z^2-(a+2i)z+b(1+i)=0$ is $2-i$, where $a, b \in \mathbf{R}$.
Find the value of a, the value of b and the other root.

29. The equation $z^2-2(1-i)z+2(2-i)=0$ has roots α and β.
Evaluate: **(i)** $\alpha+\beta$ **(ii)** $\alpha\beta$ **(iii)** $\alpha^2+\beta^2$.
Construct a quadratic equation with roots $\alpha+i$ and $\beta+i$.

Square Roots and Quadratic Equations with Complex Roots

In some problems we have to find the square root of a complex number in order to find the roots of a quadratic equation.

Example

(i) Express $\sqrt{5+12i}$ in the form $a+bi$, $a, b \in \mathbf{R}$.
(ii) Hence, determine the two roots of the equation $z^2-(1+4i)z-5-i=0$.

Solution:

(i) Let $a+bi=\sqrt{5+12i}$, $a, b \in \mathbf{R}$.

$(a+bi)^2=(\sqrt{5+12i})^2$ [square both sides]

$a^2+2abi-b^2=5+12i$ [remove brackets]

R I R R I [identify real and imaginary parts]

Real parts = Real parts

$a^2-b^2=5$ ①

Imaginary parts = Imaginary parts

$2ab=12$ ②

Solve between equations ① and ②:

$2ab=12$ ②

$ab=6$

$b=\left(\frac{6}{a}\right)$

put this into equation ①

$a^2-b^2=5$ ①

$a^2-\left(\frac{6}{a}\right)^2=5$ $\left[\text{replace } b \text{ with } \frac{6}{a}\right]$

$a^2-\frac{36}{a^2}=5$

$a^4-36=5a^2$

$a^4-5a^2-36=0$

$$(a^2-9)(a^2+4)=0$$
$$a^2-9=0 \quad \text{or} \quad a^2+4=0$$
$$a^2=9 \quad \text{or} \quad a^2=-4$$
$$a=\pm 3 \quad \text{or} \quad a=\pm 2i$$

As $a, b \in \mathbf{R}$, the result $a=\pm 2i$ is rejected.

$$b=\frac{6}{a}$$

$a=3$	$a=-3$
$b=\frac{6}{3}$	$b=\frac{6}{-3}$
$=2$	$=-2$
$a=3, \quad b=2$	$a=-3, \quad b=-2$

Thus, $\sqrt{5+12i}=3+2i$

or $\sqrt{5+12i}=-3-2i$

[i.e. $\pm(3+2i)$]

(ii) $z^2-(1+4i)z-5-i=0$

$$z^2+(-1-4i)z+(-5-i)=0 \qquad \text{[write in the form } az^2+bz+c=0\text{]}$$

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-(-1-4i)\pm\sqrt{(-1-4i)^2-4(1)(-5-i)}}{2(1)}$$
$$=\frac{1+4i\pm\sqrt{1+8i-16+20+4i}}{2}$$
$$=\frac{1+4i\pm\sqrt{5+12i}}{2}$$
$$=\frac{1+4i\pm(3+2i)}{2} \qquad \text{[put in } (3+2i) \text{ for } \sqrt{5+12i}\text{]}$$
$$z_1=\frac{1+4i+(3+2i)}{2}=\frac{1+4i+3+2i}{2}=\frac{4+6i}{2}=2+3i$$
$$z_2=\frac{1+4i-(3+2i)}{2}=\frac{1+4i-3-2i}{2}=\frac{-2+2i}{2}=-1+i$$

Thus the roots of $z^2-(1+4i)z-5-i=0$ are $2+3i$ and $-1+i$.

Note: It makes no difference if we substitute $3+2i$ or $-3-2i$ for $\sqrt{5+12i}$.

Note: Another way of asking for $\sqrt{5+12i}$ to be expressed in the form $a+bi$ is:

1. If $(a+bi)^2=5+12i$, find the values of a and b, $\quad a, b \in \mathbf{R}$.

2. If $z^2=5+12i$ or $z=\sqrt{5+12i}$, find all the values of z.

Exercise 2.6

1. Find two complex numbers $a+bi$ such that $(a+bi)^2=-15+8i$, $a, b \in \mathbf{R}$.

Express each of the following in the form $a+bi$, $a, b \in \mathbf{R}$ and $i^2=-1$:

2. $\sqrt{-3-4i}$
3. $\sqrt{8-6i}$
4. $\sqrt{-5+12i}$
5. $\sqrt{15-8i}$
6. $\sqrt{-21+20i}$
7. $\sqrt{-7-24i}$
8. $\sqrt{-9-40i}$
9. $\sqrt{2i}$

10. Express $\sqrt{-3+4i}$ in the form $a+bi$, $a, b \in \mathbf{R}$ and $i^2=-1$.
Hence, solve the equation $z^2-3z+(3-i)=0$.

11. Express $\sqrt{-24+10i}$ in the form $a+bi$, $a, b \in \mathbf{R}$ and $i^2=-1$.
Hence, solve the equation $z^2-3(1+i)z+2(3+i)=0$.

12. Express $\sqrt{15+8i}$ in the form $p+qi$, $p, q \in \mathbf{R}$.
Hence, show that one of the roots of the equation $z^2+(2+i)z-(3+i)=0$ is real and the other is complex.

13. Express $\sqrt{3+4i}$ in the form $a+bi$, $a, b \in \mathbf{R}$ and $i^2=-1$.
Hence, show that one root of the equation $z^2+(2-i)z-2i=0$ is real and the other is complex.

14. Determine real numbers x and y such that $(x+yi)^2=5-12i$.
Hence, determine the two roots of the equations:
(i) $z^2+z+(3i-1)=0$ **(ii)** $z^2-2(1+i)z+5(1-2i)=0$.

15. If $z^2=2-2\sqrt{3}i$, express z in the form $a+bi$, $a, b \in \mathbf{R}$ and $i^2=-1$.

Cubic Equations with Complex Roots

The Conjugate Root Theorem also applies to cubic equations.

Conjugate Root Theorem

> If all the coefficients of a polynomial equation are **real**, then all complex roots occur as conjugate pairs.

In other words, if one root is a complex number then its conjugate is also a root, provided all the coefficients are real.

Example

Show that $-1+2i$ is a root of $z^3-2z^2-3z-20=0$ and find the other two roots.

Solution:

Put in $(-1+2i)$ for z:

$z^3-2z^2-3z-20$

$(-1+2i)^3-2(-1+2i)^2-3(-1+2i)-20$

$=(11-2i)-2(-3-4i)-3(-1+2i)-20$

$=11-2i+6+8i+3-6i-20$

$=20-20+8i-8i$

$=0$

$\therefore\ -1+2i$ is a root.

$(-1+2i)^2$
$=(-1+2i)(-1+2i)$
$=-3-4i$

$(-1+2i)^3$
$=(-1+2i)^2(-1+2i)$
$=(-3-4i)(-1+2i)$
$=11-2i$

$\therefore\ -1-2i$ is also a root (roots occur in conjugate pairs, as all the coefficients are real).

We now construct the quadratic factor using:

$z^2-(\text{sum of the roots})z+(\text{product of the roots})$

$z^2-(-1+2i-1-2i)z+(-1+2i)(-1-2i)$

z^2+2z+5

Division:

$$
\begin{array}{r|l}
 & z-4 \\
z^2+2z+5 & z^3-2z^2-3z-20 \\
 & z^3+2z^2+5z \\
 & -4z^2-8z-20 \\
 & -4z^2-8z-20 \\
 & 0
\end{array}
$$

Dividing $z^3-2z^2-3z-20$ by z^2+2z+5 gives $z-4$.

$\therefore$ the third factor is $z-4$.

Let $z-4=0$

$\therefore\quad z=4$

Thus, the other two roots are $-1-2i$ and 4.

Note: We did not have to use long division to get the third factor.
Let the third factor be $z+k$.

$\therefore\quad (z+k)(z^2+2z+5)=z^3-2z^2-3z-20$

Comparing constants on both sides, we get:

$5k=-20$

$\therefore\quad k=-4$

Thus, the third factor is $z-4$.

Example

One root of the equation $z^3 + az^2 + bz - 52 = 0$, $a, b \in \mathbf{R}$ is $2 - 3i$.
Find the value of a and the value of b.

Solution:

Method 1:

$(2-3i)^2 = (2-3i)(2-3i) = -5-12i$

$(2-3i)^3 = (2-3i)^2(2-3i) = (-5-12i)(2-3i) = -46-9i$

$$z^3 + az^2 + bz - 52 = 0$$

$$(2-3i)^3 + a(2-3i)^2 + b(2-3i) - 52 = 0 \qquad \text{(put in } (2-3i) \text{ for } z)$$

$$(-46-9i) + a(-5-12i) + b(2-3i) - 52 = 0$$

$$-46 - 9i - 5a - 12ai + 2b - 3bi - 52 = 0$$

$$(-5a + 2b - 98) + (-12a - 3b - 9)i = 0 \qquad \text{(group real and imaginary parts together)}$$

$\therefore \quad -5a + 2b - 98 = 0$ ① and $-12a - 3b - 9 = 0$ ②

Solving the simultaneous equations ① and ② gives $a = -8$ and $b = 29$.

Method 2:

If $2 - 3i$ is a root, then $2 + 3i$ is also a root.
(Roots occur in conjugate pairs as all the coefficients are real.)
The quadratic factor is given by:

$z^2 -$ (sum of the roots)$z +$ (product of the roots)

$z^2 - (2 - 3i + 2 + 3i)z + (2-3i)(2+3i)$

$z^2 - 4z + 13$

Let the third factor be $z + k$.

$$(z+k)(z^2 - 4z + 13) = z(z^2 - 4z + 13) + k(z^2 - 4z + 13)$$

$$= z^3 - 4z^2 + 13z + kz^2 - 4kz + 13k$$

$$= z^3 + (-4+k)z^2 + (13-4k)z + 13k$$

$\therefore \quad z^3 + (-4+k)z^2 + (13-4k)z + 13k = z^3 + az^2 + bz - 52$

Equating coefficients of like terms:

$-4 + k = a$ ① $\qquad 13 - 4k = b$ ② $\qquad 13k = -52$ ③

$13k = -52$ ③

$\therefore \quad k = -4$ (replace k with -4 in ① and ②).

$-4 + k = a$ ①

$-4 - 4 = a$

$-8 = a$

$13 - 4k = b$ ②

$13 - 4(-4) = b$

$13 + 16 = b$

$29 = b$

Thus, $a = -8$ and $b = 29$.